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FAST TRACK COMMUNICATION

A note on the quantum-tail effect on fusion reaction rate

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Abstract

A study is made of the power-law tail effect in the quantum particle distribution over momentum on the nuclear fusion reactions. Our results do not support the idea of averaging the fusion reaction cross-section over the momentum distribution, postulated and used in many publications.

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While in classical statistics all systems of particles in equilibrium have single-particle momentum distribution (MD), $n(\vec{p})$, of Maxwell–Boltzmann form, the MD in quantum statistics has a non-Maxwellian form, contains a non-exponential tail [1–4] and plays a role central to our understanding of interacting quantum particle systems. $n(\vec{p})$ has to satisfy the sum rules

$$\int n(\vec{p})\,\mathrm{d}^3 p = 1, \tag{1}$$

$$\frac{1}{2m}\int n(\vec{p})p^2\,\mathrm{d}^3 p = T_k, \tag{2}$$

where T_k is the kinetic energy.

In order for the kinetic energy integral, equation (2), to remain finite, the MD should decline faster than p^{-5}. The large p-behavior of $n(\vec{p})$ has been considered in many papers [1–4]. It was shown that the interaction between quantum particles leads to the appearance of a power-law tail in the MD. It was found for the first time in [2] that the zero temperature MD for an interacting electron gas should go as $1/p^8$ for large p.

A general expression for $n(\vec{p})$ can be written as [5]

$$n(\vec{p}) = \int f_\gamma(E,\vec{p})\,\mathrm{d}E, \tag{3}$$

where the energy–momentum distribution $f_\gamma(E,\vec{p})$ can be represented as [5]

$$f_\gamma(E,\vec{p}) = f(E)\delta_\gamma(E,\vec{p}), \tag{4}$$

$f(E)$ is the occupation number (Fermi–Dirac, Bose–Einstein or Maxwell–Boltzmann), $\delta_\gamma(E, \vec{p})$ is the spectral function [5]

$$\delta_\gamma(E, \vec{p}) = \frac{\gamma(E, \vec{p})}{\pi[(E - \epsilon_p - \Delta(E, \vec{p}))^2 + \gamma^2(E, \vec{p})]}, \tag{5}$$

$\epsilon_p = p^2/(2m)$, and $\gamma(E, \vec{p})$ and $\Delta(E, \vec{p})$ are solutions of 'complex set of integral equations' [5].

The finite value of $\gamma(E, \vec{p})$ leads to the appearance of power-law tails in the MD.

After integration over momentum of $f_\gamma(E, \vec{p})$, it is easy to obtain that the energy distribution

$$n_E(E) = \int f_\gamma(E, \vec{p})\, \mathrm{d}^3 p \tag{6}$$

remains exponential [6]. The difference between the two distributions, equations (3) and (6) is related to the quantum uncertainty.

References [6, 7] have been suggested to average reaction rates over $n(\vec{p})$ rather than distribution over energy $n_E(E)$ (see also [8]). Since (i) in this case quantum tails might produce dramatic effects on the rates of nuclear and other reactions in a medium and (ii) the provocative suggestion [6, 7] has been used in a many papers [6, 7, 9–18], it is the purpose of the present work to examine the validity of the approximation [6, 7].

We stress here that directly measured cross sections of low-energy nuclear reactions in a medium, when available, are higher than the expected values [19–32]. To date, the observed enhancement factors are not completely understood [19]. An example of the effort where environment considerations have been carried out for low-energy processes are [33–35] (see also [36, 37] where the tunneling of bound two-body systems through a potential barrier have been considered).

Let us first study the effect of the binding of the deuteron inside the hydrogen atom on the example of the p–d fusion reaction. The bound deuteron does not have a definite velocity. From its wavefunction one can calculate the deuteron velocity distribution

$$n(\vec{v}_d) = \frac{8}{\pi^2 v_0^3} \frac{1}{(1 + (v_d/v_0)^2)^4}, \tag{7}$$

where $v_0 = (m_e/m_d)e^2/\hbar$, and m_e and m_d are the electron mass and deuteron mas, respectively. One has

$$\int n(\vec{v}_d)\, \mathrm{d}^3 v_d = 1. \tag{8}$$

Now, the bound deuteron and the proton have a relative velocity

$$v_{\mathrm{rel}} = |\vec{v}_d - \vec{v}_p| \tag{9}$$

and $\langle \sigma v \rangle$ is then

$$\langle \sigma v \rangle = \int S(E) \frac{1}{E} \exp\left(-\pi \sqrt{\frac{E_G}{E}}\right) v_{\mathrm{rel}} n(\vec{v}_d)\, \mathrm{d}^3 v_d, \tag{10}$$

where $E = \mu v_{\mathrm{rel}}^2/2$, μ is the reduced proton deuteron mass, $S(E)$ is the astrophysical S-factor and $E_G = 2e^4\mu/\hbar^2$.

In the case of small v_p ($v_p < v_0$) the $\langle \sigma v \rangle$ value, equation (10) corresponds to the screening energy $E_{\mathrm{scr}} \approx 300$ eV. This result is not in agreement with three-body adiabatic calculations [8, 38] (more than 10 times larger). In our future work we will consider an application of the Sturm-function method in the formalism of the three-body Faddeev–Hahn equation [39, 40] for this problem.

We consider a system of N identical quantum particles carrying the unit positive charge, e, and contained in a volume Ω (periodic boundary conditions) with an uniform external background field of the opposite sign which neutralizes the total charge of the system. The time independent nth state wavefunctions of the system $\Phi_n(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ with the energy E_n are assumed to be normalized.

At a thermal equilibrium, $n(\vec{p})$ can be written as

$$n(\vec{p}) = \frac{\sum_n f_n(\vec{p})\, \mathrm{e}^{-E_n/(k_B T)}}{\sum_n \mathrm{e}^{-E_n/(k_B T)}}, \tag{11}$$

where $f_n(\vec{p})$ is the probability to find a particle with the momentum p in the nth state Φ_n

$$f_n(\vec{p}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\vec{p}(\vec{r}-\vec{r}')} \Phi_n^*(\vec{r}, \vec{r}_2, \ldots, \vec{r}_N) \Phi_n(\vec{r}', \vec{r}_2, \ldots, \vec{r}_N)\, \mathrm{d}^3 r\, \mathrm{d}^3 r' \prod_{i=2}^{N} \mathrm{d}^3 r_i. \tag{12}$$

A generalization of the Kimball method, [1], leads to the following large-p behavior of $f_n(\vec{p})$

$$\lim_{p\to\infty} f_n(\vec{p}) = |\psi_n(0)|^2 \frac{2}{\pi} \frac{N-1}{\Omega} \hbar \frac{m^2 e^4}{p^8}, \tag{13}$$

where

$$|\psi_n(0)|^2 = \Omega \int |\Phi_n(\vec{r}, \vec{r}, \vec{r}_3, \ldots, \vec{r}_N)|^2\, \mathrm{d}^3 r \prod_{i=3}^{N} \mathrm{d}^3 r_i \tag{14}$$

and m is the particle mass. Substituting the large p asymptotic, equation (13), into equation (11) we find the large momentum tale of the momentum distribution $n(\vec{p})$ at the temperature T

$$\lim_{p\to\infty} n(\vec{p}) = \frac{2}{\pi} \hbar \rho \frac{m^2 e^4}{p^8} |\Psi(0)|^2, \tag{15}$$

where

$$|\Psi(0)|^2 = \frac{\sum_n |\psi_n(0)|^2\, \mathrm{e}^{-E_n/(k_B T)}}{\sum_n \mathrm{e}^{-E_n/(k_B T)}} = \frac{\pi}{2\hbar\rho m^2 e^4} \lim_{p\to\infty} n(\vec{p}) p^8 \tag{16}$$

is the contact probability of finding two particles at zero separation, and $\rho = (N-1)/\Omega$ is the density for $N \gg 1$.

We thus see that the large-p behavior is governed by the contact probability of finding two particles at short distance. Although, the power-law tail in the momentum distribution was observed in many papers (see, for example [1–4]), it was shown for the first time in [1] that for the ground-state coefficient of the $1/p^8$ term is proportional to the zero separation probability $|\psi_0(0)|^2$.

Now, we consider the nuclear reaction between nuclei, i and j, under conditions that exist in stellar interiors. The nuclear cross-section, σ_{ij} for small collision speed is approximated by

$$\sigma_{ij}(E) = \frac{S_{ij}(E)}{E} \exp(-\pi\sqrt{E_G/E}) \approx \frac{S_{ij}(0)}{E} \exp(-\pi\sqrt{E_G/E}), \tag{17}$$

where a slowly varying function with E $(E \ll E_G)$ $S_{ij}(E)$ is called the astrophysical S-factor, $E_G = 2\mu_{ij} Z_i^2 Z_j^2 e^4/\hbar^2$ is the Gamov energy, $\mu_{ij} = m_i m_j/(m_i + m_j)$ and Z denote charge numbers. The last equation (17) is practically exact—for example, the error in the replacement $S(E)$ by $S(0)$ for the proton–proton fusion, $p + p \to^2 D + e^+ + \nu_e$ is only about 0.5% [41] for energies corresponding to temperatures in the center of the sun.

The number of reactions between nuclei of i and j species at number densities ρ_i and ρ_j with a relative kinetic energy E is calculated as

$$R_{ij} = \frac{\rho_i \rho_j}{\delta_{ij} + 1} \sigma_{ij}(E) v_{ij}, \tag{18}$$

where $v_{ij} = \sqrt{2E/\mu_{ij}}$.

The Gamov rate is calculated by averaging equation (18) with the Maxwell–Boltzmann distribution $f_{M-B}(E)$ at temperature $T \ll E_G/k_B$

$$f_{M-B}(E) = \frac{2}{k_b T}\sqrt{\frac{E}{\pi k_B T}} \exp\left(-\frac{E}{k_B T}\right). \tag{19}$$

The result yields the Gamov rate [42]

$$R^G_{ij} = \frac{\rho_i \rho_j}{\delta_{ij} + 1} \langle \sigma_{ij} v_{ij} \rangle_{M-B} = \frac{\rho_i \rho_j}{\delta_{ij} + 1} \int f_{M-B}(E) \sigma_{ij}(E) v_{ij} \, \mathrm{d}E. \tag{20}$$

It is possible to rewrite the rate in terms of the contact probability (the square of the wavefunction at the origin [43])

$$R_{ij}(E) = \hbar S_{ij}(E) \frac{\rho_i \rho_j}{\pi(1 + \delta_{ij})\mu_{ij} Z_i Z_j e^2} |\psi_{ij}(0)|^2, \tag{21}$$

where ψ_{ij} is the Coulomb wavefunction with a normalization such that

$$\int_\Omega |\psi_{ij}(\vec{r})|^2 \, \mathrm{d}^3 r = \Omega \tag{22}$$

over a large volume Ω [43]. Indeed, the square of the wavefunction at the origin then takes on value

$$|\psi_{ij}(0)|^2 = \frac{\pi\sqrt{E_G/E}}{\exp(\pi\sqrt{E_G/E}) - 1}. \tag{23}$$

For the N-body system, equations (11)–(16), the number of binary fusion reactions per unit time and unit volume in the nth state $\Phi_n(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ is

$$R_n = S(0)\hbar \frac{\rho^2}{\pi m e^2} |\psi_n(0)|^2 = S(0) \frac{\rho}{2(me^2)^3} \lim_{p\to\infty} (p^8 f_n(\vec{p})). \tag{24}$$

In (24) $|\psi_n(0)|^2$ and $f_n(\vec{p})$ are given by (14) and (13), respectively.

At the thermal equilibrium

$$R = \frac{\sum_n R_n \mathrm{e}^{-E_n/(k_B T)}}{\sum_n \mathrm{e}^{-E_n/(k_B T)}} = S(0) \frac{\rho}{2(me^2)^3} \lim_{p\to\infty} (p^8 n(\vec{p})). \tag{25}$$

Now, we want to see if the rate R, equation (25), and the rate R_Q, calculated in [15] by averaging the pp-fusion reaction cross-section over quantum momentum distribution [3], stand in contradiction to each other for the pp fusion in the star. To do so we note that, in the Galitskii–Yakimets approximation for $n(\vec{p})$ at large momentum [3], ratio R/R_Q is

$$\frac{R}{R_Q} \approx 10^3,$$

where the standard solar model parameters [44] are used.

Clearly, the error in averaging the fusion reaction cross-section over quantum momentum distribution is that one has neglected the coupling between the various probability amplitudes of velocity which is introduced by the quantum uncertainty.

In conclusion, we summarize the main points of this communication.

(i) We have considered nuclear motion inside the atom and have found that for the p–d fusion in the case of small v_p, the $\langle \sigma v \rangle$ value is not in agreement with three-body adiabatic calculations [8, 38].

(ii) For the *N*-body system, equations (11)–(16), we have found a general expression for calculating the nuclear fusion rate at thermal equilibrium, equation (25).

(iii) Our results do not support the idea of averaging the fusion reaction cross-section over the momentum distribution, postulated in [6, 7] and used in many publications [6, 7, 9–18].

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